What Shape is a Tree?

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Abstract

What shapes can geometric trees make? By varying the common ratio between levels, the angle between branches, and the branching factor, one can generate a surprising number of shapes — the simplest being regular polygons and various grids. Patterns with fractal-like margins expectedly appear. Stars appear! The seemingly solid patterns have complex inner structures. We end with a challenge.

This short talk was based on the "Geom-e-Trees" poster designed by the author.

A binary tree has two branches at every node, a ternary tree has three, and so on. We could refer to this as the tree's "naryness" or "branching factor".

A *level* inside the tree is the branch distance away from the root of the tree.

Geometric trees are drawn using a common angle between branches throughout the tree and a common ratio between one level and the next. I find the term "reduction factor" less ambiguous than "common ratio", but use both terms. When generating a tree, if the reduction factor is too "strong" the tree's growth will be stunted. If the reduction factor is too weak, the tree will overlap a lot, making it look like a real tree -- perhaps a bit on the wild side. But if reduction factor is *just right* you get what Mandlebrot called a self-touching tree. The "perfect" reduction factor turns out to be a function of the angle *and* degree of tree -- but that is another problem [2].

Self-Touching Trees. The Geom-e-Trees [1] poster has three tiers. The top tier (Figure 1.) shows selected self-touching trees, with the angle increasing to the right in each column, and naryness varying from 2 to 6 on the rows. Note that each tree has been scaled to fill its cell in the table.

Equi-Length Trees. In the middle tier of the poster (Figure 2.), the common ratio is 1.o. This means the branches in each tree are equi-length. The trees are all essentially "unit distance graphs". What happens here is that the 2, 4 and 6-branched trees snap into a hexagonal grid at 120 degrees -- the branches are confined to the grid. Likewise, 3 and 5-branched trees make a diamond-shaped square grid at 90 degrees, but make a hexagonal-shaped isometric grid at 60 and a triangle-shaped isometric grid at 120 degrees... due to the same grid confinement of the branches at those angles.

I wonder:

How do all the branches stack up on the grid lines?
How does the naryness (branching factor) of a tree affect the resulting grid?
Would these grids have more cells if the tree had more levels?

Doubling Trees. In the bottom tier of the poster (Figure 3.), the common ratio is 1/2, meaning that branches double in length from one level to the next — they really go wild... polygons, pinwheels, and stars.

The square grids we saw above become area-filling diamonds, and the different isometric grids morph into either a hexagon or a triangle. (Compare the 3-ary and 5-ary versions in figures 2 & 3.)

The hexagonal grids become really *weird* — downright scary.

In the spirit of MG, I leave you with a challenge. Start with the christmas-tree-like Ternary tree and make its common angle 144 degrees, then reduce the common ratio down to 0.5... and we get... a FIVE-pointed STAR! *Explain how the star pattern is formed.*

By the way, the more levels you add to the tree, the more filled in the ternary star gets. As you can see in Figure 4, the Heart of the Star is a very busy place.

Summary. We only looked at a few of the 150 trees on the poster, and the poster represents only a sample of all Geom-e-Trees. I discovered these special trees as I was developing my iPad/iPhone app "Geom-e-Tree" [3]. The app provides an immersive experience of the geometric tree space.

Geom-e-Tree can handle all reduction factors above 0.5, and all integral angles from 0 to 359. Check it out! A children's version, "Geom-e-Twee", is free for all iPhones and iPads. RT Please!

Geom-e-Tree

This paper is meant to supplement a short talk at Gathering for Gardner 10 in Atlanta, March, 2012. There were ~20 slides in the presentation, some juxtaposing trees from the poster [1].

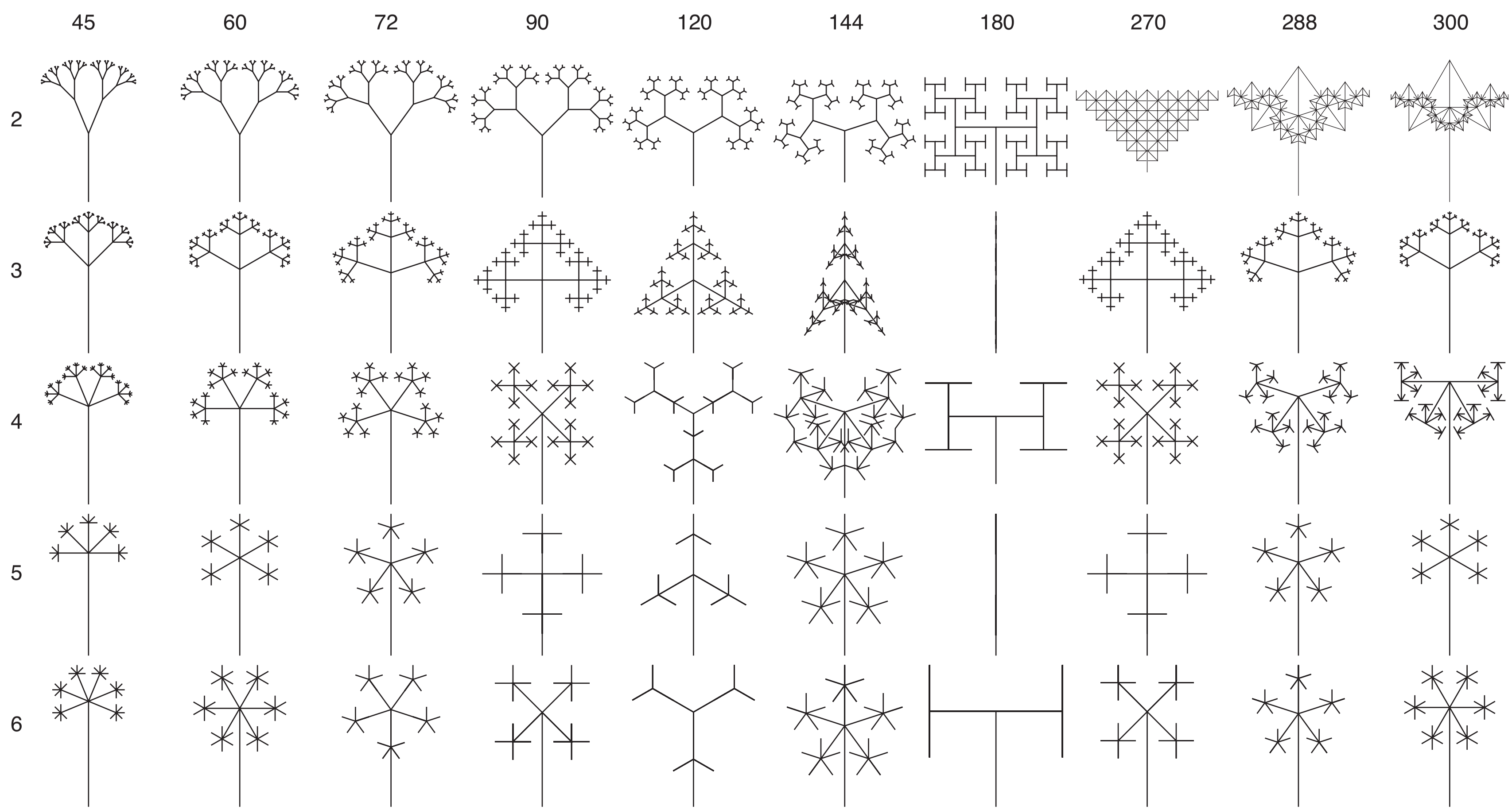

Figure 1. Self-Contacting Trees

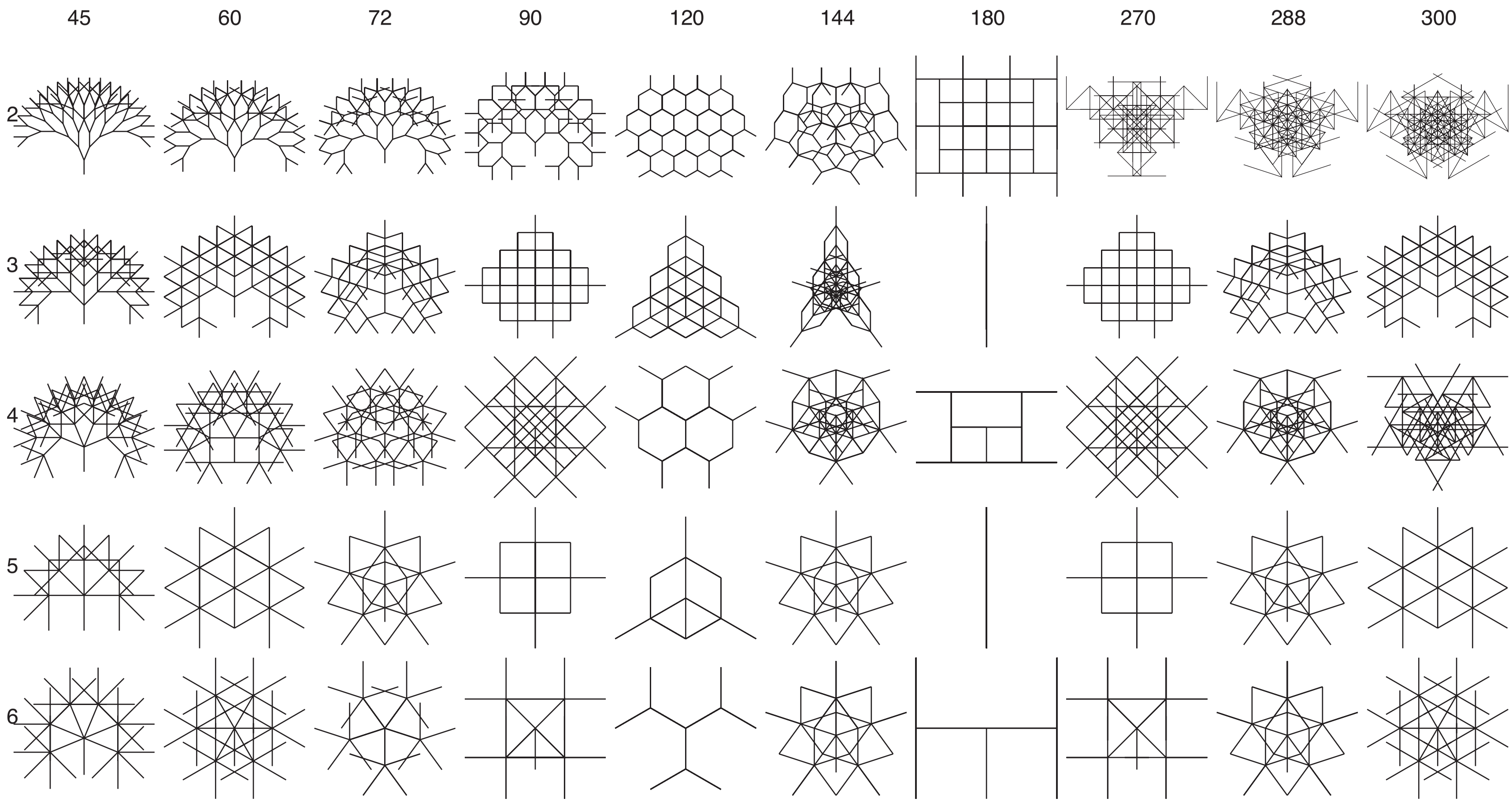

Figure 2. Unit Distance Trees

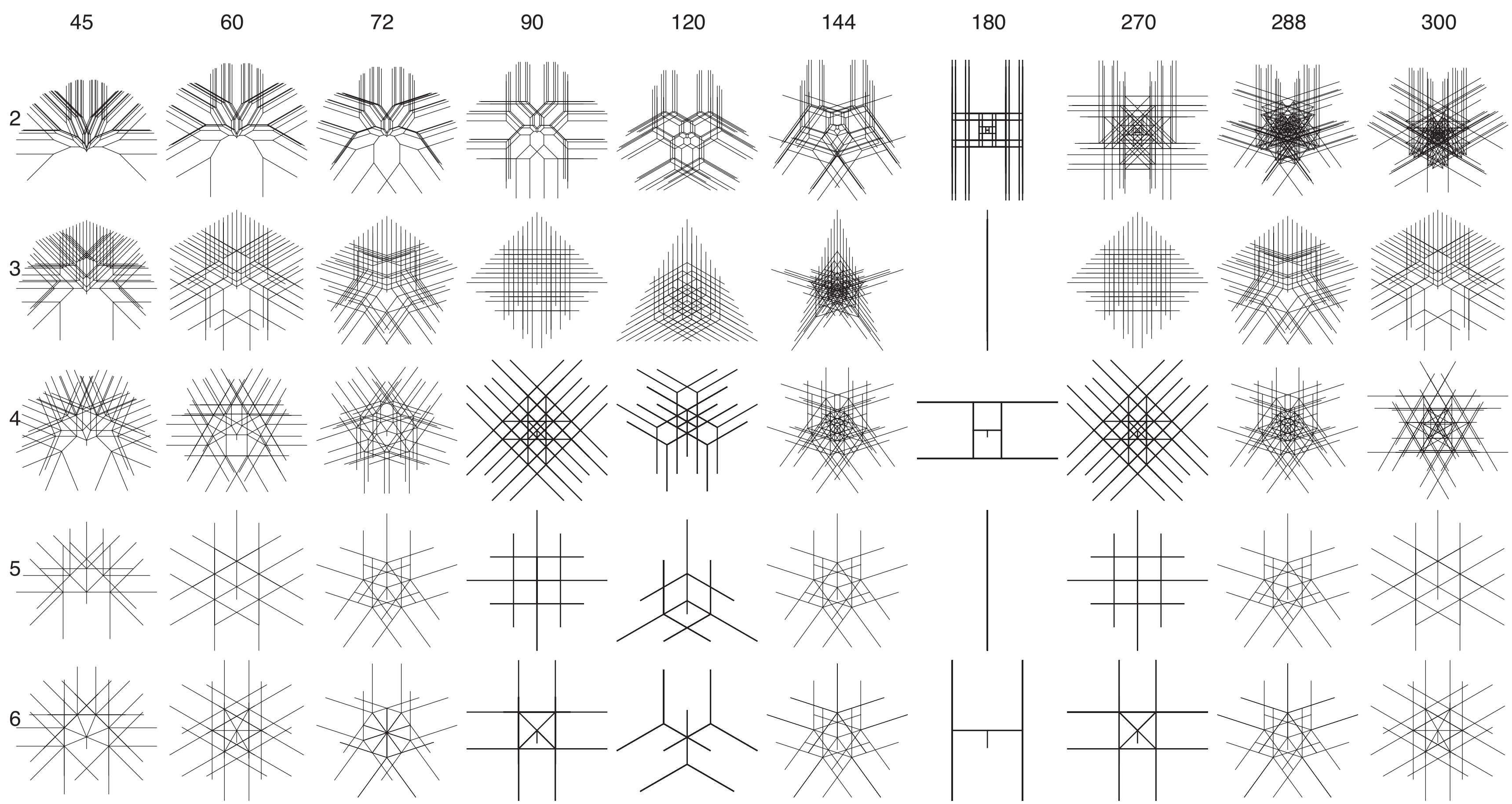

Figure 3. Doubling Trees

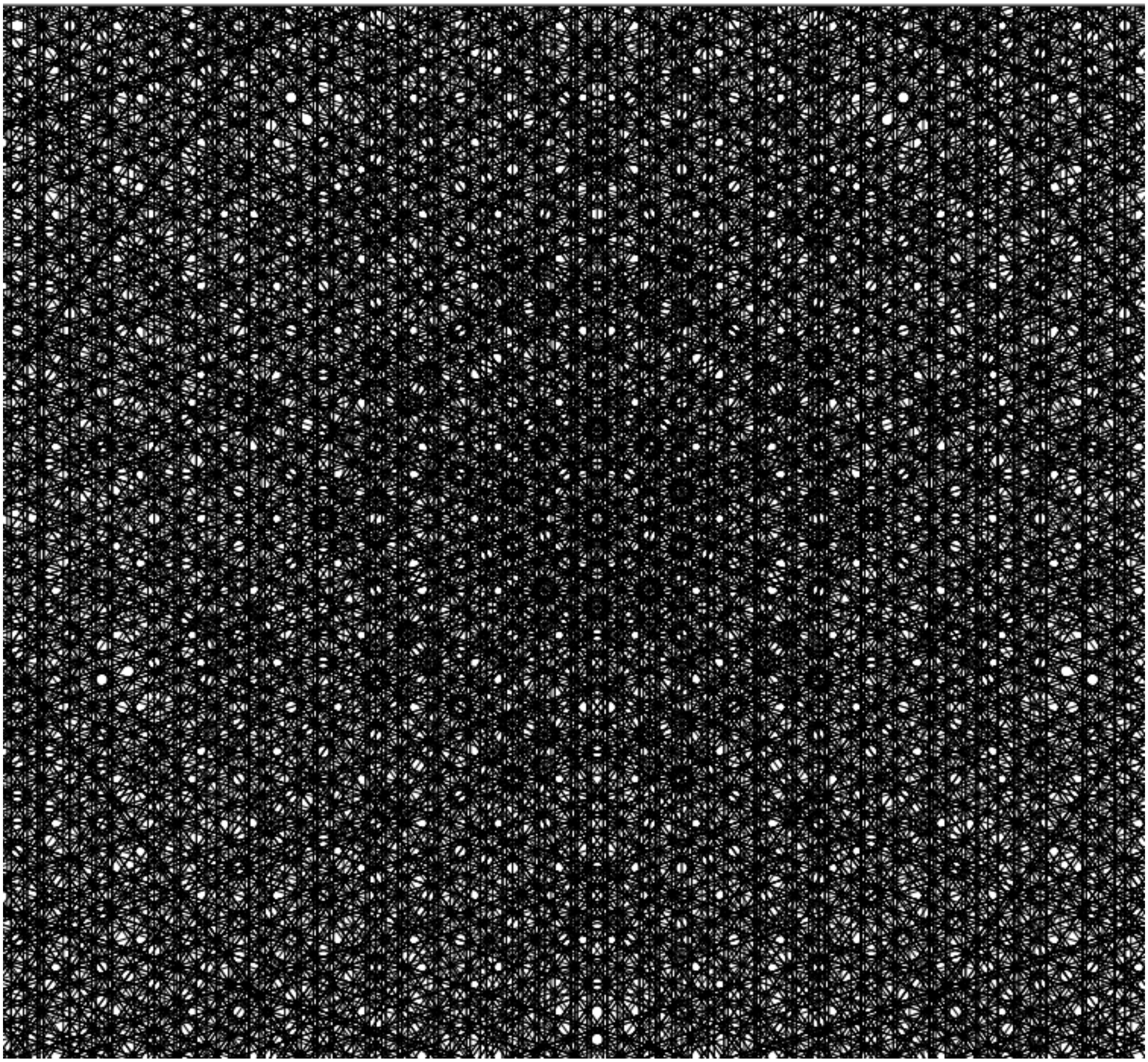

Figure 4. The Heart of the Ternary Star (cropped from a larger image)

References

[1] http://geom-e-tree.com/poster.html (link to downloadable 11x17 poster).
[2] http://faculty.plattsburgh.edu/don.west/trees/ - Self-Contacting Fractal Trees, Don West.
[3] http://geom-e-tree.com - commercial web site for Geom-e-Tree.