Langford's Problem, Remixed

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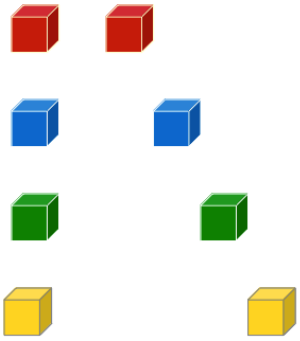
Abstract

Martin Gardner introduced Langford's Problem to his readers in Mathematical Games in November 1967. This paper is a summary of work that has been done to date. We also look at some variants and related artwork and puzzles.

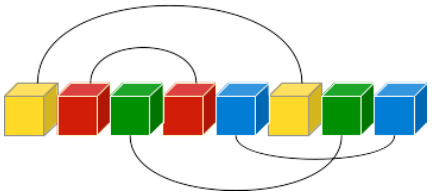
We are given a number of PAIRS of unit-sized blocks...

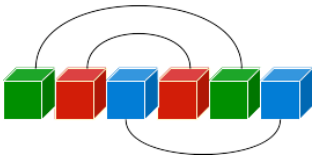
We add a Constraint to each pair: one pair of blocks is ONE unit apart, another pair is TWO units apart, and so on. The pairs are *entangled*. If you move one block, the other block keeps its distance.



Can we fit 4 pairs into 8 slots?

Yes. 4 pairs fit nicely, as do 3 pairs.





The arcs just help show the pairings.

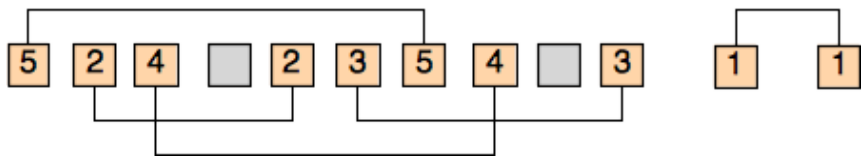
Substituting numbers for colors, C. Dudley Langford concluded, in 1958:

"Clearly this is a perfectly general problem."

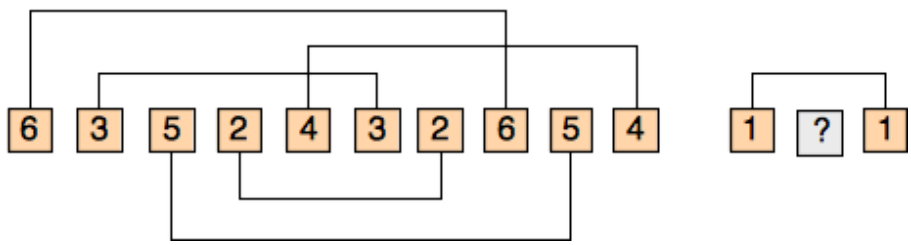
Langford found arrangements for 7, 8, 11, 12, and 15 pairs...

```
7      7 3 6 2 5 3 2 4 7 6 5 1 4 1
8      8 3 7 2 6 3 2 4 5 8 7 6 4 1 5 1
11     10 8 9 5 2 11 7 2 6 5 8 10 9 4 7 6 3 11 4 1 3 1          <======  11 !
12     10 8 9 2 4 12 2 11 7 4 8 10 9 3 5 6 7 3 12 11 5 1 6 1
15     12 10 8 14 5 3 1 15 1 3 5 8 10 12 7 13 11 9 14 6 4 2 7 15 2 4 6 9 11 13
```

But he couldn't find an arrangement of 5 or 6 pairs, or any of the others < 15. No matter how hard one tries you can't fit the last pair in:



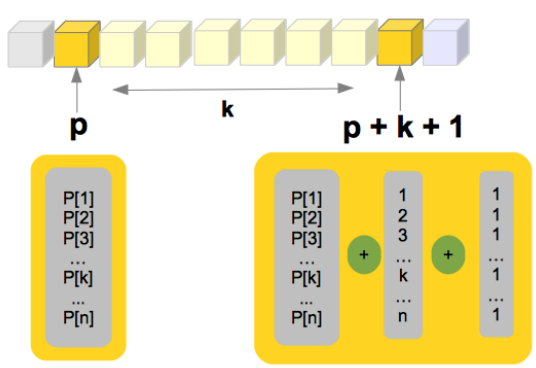
Or you can fit them together, but have a pair left over:



Langford wanted to know when a certain number "N" clicked. Roy Davies and others figured out why. The trick is to set up two different summations of the same set of numbers and see what shakes out. The first summation is simply the available positions for the arrangement. These would be the numbers 1 to 2n, which is this simple summation:

$$\sum_{i=1}^{2n} i = \frac{2n(2n+1)}{2} = n(2n+1) = 2n^2+n$$

The second summation uses our constraint that blocks have *pairs* of positions. And those *pairs* of positions differ by the unique distances 1 thru n.



Here we have the summation of the paired positions:

$$\underbrace{\sum_{i=1}^{n} p_i + \sum_{i=1}^{n} p_i}_{2\sum p_i} + \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$2\sum p_i + \frac{n(n+1)}{2} + n$$

In a perfect arrangement, those pairs fill the available positions... So, we can equate the two sums, and with a wave of the hand, and voila! We see 'n' must be a multiple of 4 or one less:

$$2\sum_{i=1}^{n} p_i + \frac{n^2+n}{2} + n = 2n^2 + n$$

$$2\sum_{i=1}^{n} p_i + \frac{n^2+n}{2} = 2n^2$$

$$4\sum_{i=1}^{n} p_i + n^2+n = 4n^2$$

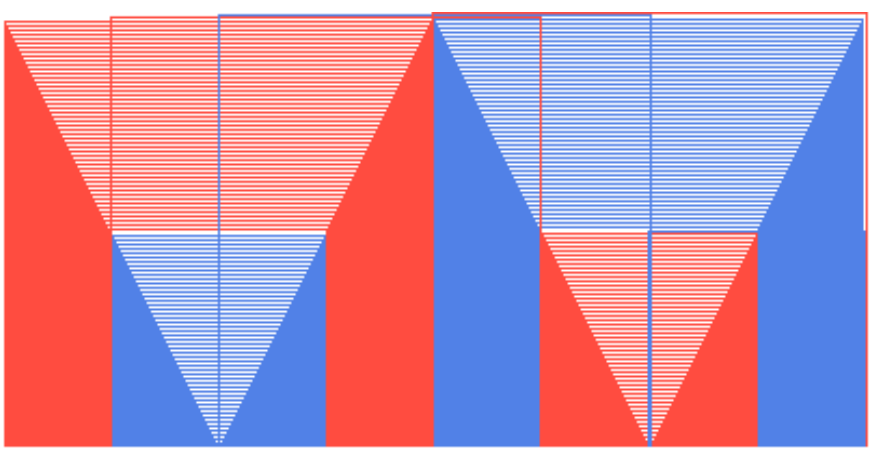
$$4\sum_{i=1}^{n} p_i = 4n^2 - n^2 - n$$

$$4\sum_{i=1}^{n} p_i = 3n^2 - n$$

$$\text{integer} \rightarrow \sum_{i=1}^{n} p_i = \frac{3n^2-n}{4}$$

$$\boxed{\therefore\ n \sim 4m \text{ or } 4m-1}$$

Any multiple of 4 or one less, like say 200? Roy O. Davies defined a method to construct a simple arrangement for a given n. This shows 200 pairs.



Davies nests odd & even, and then bridges between them with a couple pairs. You can come up with other arrangements, but this construction always works.

If there are multiple arrangements... how many are there for .. 11 for example? This is where I came into Langford's Problem, when Martin crafted one of his Brain Teasers in November 1967. People got on their big computers and shifted bits around or digits around and came up with some answers.

```
7  -> 26
8  -> 150
11 -> 17,792     <===  There's the answer for n=11 !
12 -> 108,144
15 -> 39,809,640
16 -> 326,721,800
19 -> 256,814,891,280
```

Generations of hardware went by.... But exhaustive search method simply got too Exhaustive! So, what, maybe do "Langford at Home" in little chunks, using millions of PCs? Nah. But then came the "World Wide Web", so I posted what we had on the web and on Sloane's On-Line Integer Sequence Server. I continue to curate a set of web pages covering Langford's Problem. [1]

Godfrey's Method. In 2002, English dense matter physicist Mike Godfrey (initials: MG!) came up with an "Algebraic Method". His method evaluates lots of sums of products, and divides it all by a very large number. Godfrey and another chap computed L(2,20) = 2,636,337,861,200 (2.6 Trillion) In the process, Godfrey realized there were some shortcuts for evaluating the sums and products.

$$\sum_{(x_1,\ldots,x_{2n})\in\{-1,1\}^{2n}} \left(\prod_{i=1}^{2n} x_i\right) \prod_{i=1}^{n} \sum_{k=1}^{2n-i-1} x_k x_{k+i+1} = 2^{2n+1} L(2,n)$$

Martin asked me for a simple explanation of this. Sorry! You can find Godfrey's explanation on my web page. It's like Roy O. Davies' proof, on steroids!

French Team employs grid computing system. Word spread about Godfrey's Method. Soon, a French team, M Krajecki and Chistophe Jaillet, used lots of computers to get 23 & 24:

```
23    3,799,455,942,515,488
24    46,845,158,056,515,936  (46 quadrillion)
```

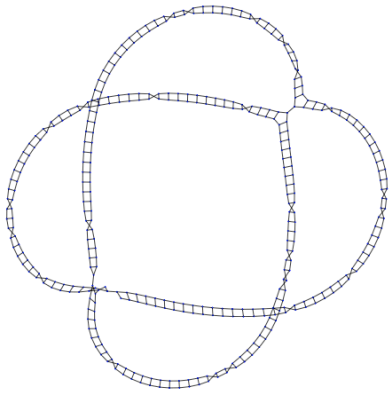
Chistophe's doctoral thesis was partly about efficient evaluation Godfrey's Method, in a distributed manner. L(2,27) is up for grabs at this point.

Odd/Even Parity as key to Solvability. An arrangement of pairs of numbers 1,1, 2,2, 3,3, ... n,n will have 2n positions numbered 1..2n. Half the positions will be oddly-numbered, the other half evenly-numbered.

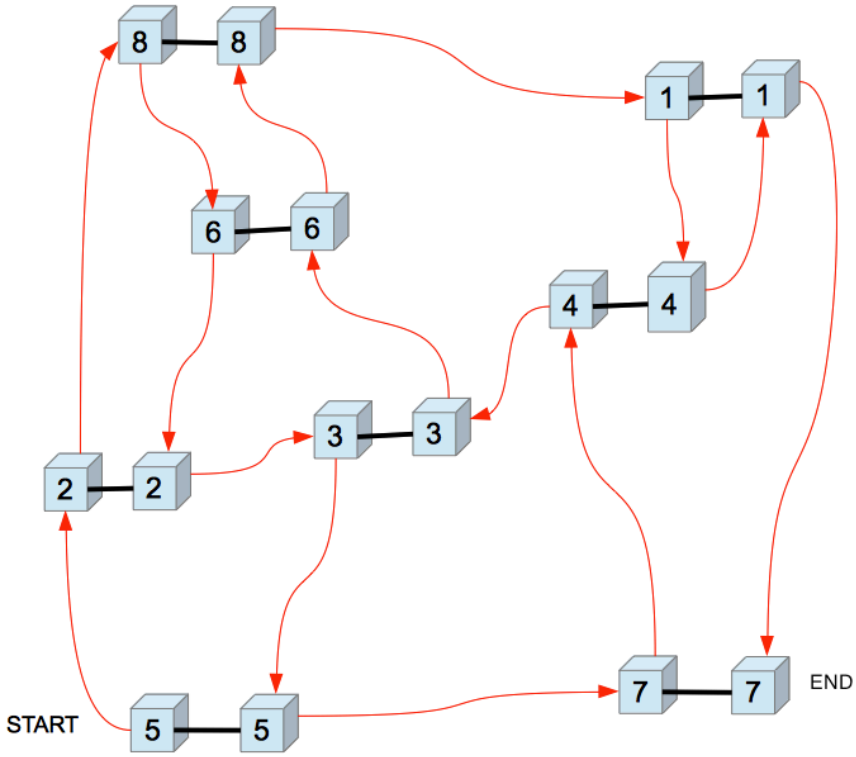
An even pair will take an odd position and an even position, no matter where the pair is in the arrangement. For example, if Left 2 is in position 1, the Right 2 must be in position 4.

However, an odd pair will take either two odd positions or two even positions. Therefore the odd pairs must occur in pairs themselves to preserve parity if there is any hope of covering all the positions in a full arrangement. So, there must be an even number of odd pairs. This happens only when n is 3, 4, 7, 8, 11, 12, 15, 16, ..., 4m-1, 4m.

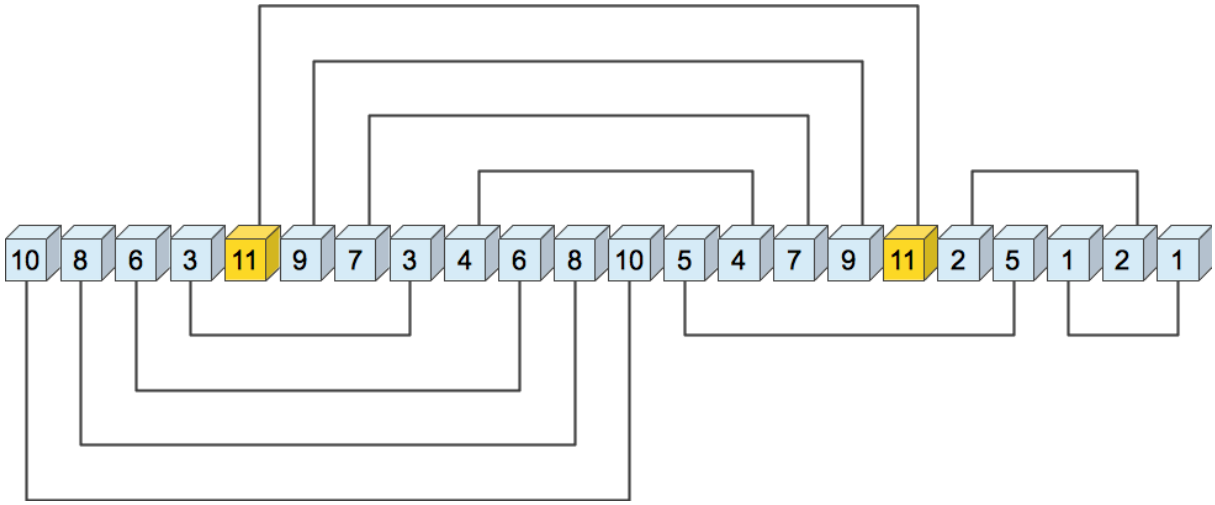
Some Fun with Langford's Problem. For this, refer back to Davies' constructed solution for 200. Here is the same arrangement with the blocks paired up across fom each other. Notice the bridges connecting the nested sets. The twists are an artifact of GraphViz's NEATO grapg layout algorithm.



The Bridges of Konigsburg? Here's a fun graph of the path through 8 pairs of blocks [5286235743681417, see planar]. Notice the cycle of 4 blocks between the two 4's, and so on. It's like taking a tour of a city, without walking over *any* of the bridges!



KNUTH's PLANAR solutions. When Donald Knuth visited Martin in NC he discovered "The Langford Files". Back home, Knuth got interested in Planar Solutions. There are far fewer planar solutions. For example, only sixteen of 17,792 solutions for n=11 are planar. Knuth enumerated planars through n=28, and registered his results on OEIS.

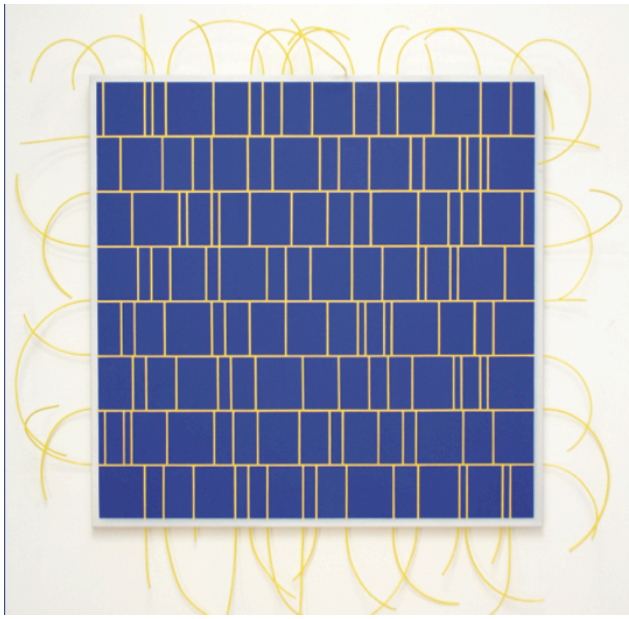


I'd like to see Knuth's results confirmed. By the way, one should define what is meant by planar in this context. No fair doing an "end run" in order to connect two blocks. They must be connected directly by going toward each other on one side or another of the arrangement. (Any example of an arrangement that would qualify otherwise? Is this a dumb statement?)

Related Keywords: Arc Diagram, Linear Encoding, Planar Graph, Golumb Ruler, Skolem, Steiner.

PUZZLES and ART

German artist Gerhard Hotter did many Langford-themed artworks over a period of years. Recently Hotter translated them into sound. You can find the Link on my website or search YouTube for Hotter.



BLUECHIP I, 2008, 120x120 cm (about 160x160 cm including the tubes)

This puzzle, by Daniel Hardisky, employs entangled pairs of wooden disks. Hardisky also did a 3D version “Devil's Gate”.

MacTutor Logo

I wrote to John O'Connor to tell him that the MacTutor logo was almost a Langford arrangement. So he updated the search form for MacTutor to randomly choose a logo, Langfordly colored. He uses both 41312432 and its reverse, shown here.

MacTutor

Charles Dudley Langford, Biography

Langford started out as an industrial chemist and was a member of Royal Chemical Society. He became a passionate Maths teacher who thought about all kinds of mathy problems, publishing 30 notes in Mathematical Gazette. He co-authored a paper on Hinged Dissections with Martyn Cundy, and corresponded with Harry Lindgren. Little did we know that Langford was dying during the time we were cranking away on his problem using computers in 1967 and 1968.

Charles Dudley Langford was born in Highgate (near London) in October 16, 1905.
He contracted polio at age 12.
At some point, he moved to Ayrshire, Scotland.
Langford died Jan 11, 1969 at age 63, in Girvan.
His (a) son signed his death certificate - C. A. Langford (Informant).
Charles Andrew Langford was born in 1938 in Girvan.
Langford is buried in Girvan, Ayrshire district of Scotland, in the East Doune (Girvan) Cemetery.

CLOSING

The "Problem" came about when a father watched his son play with colored blocks. My son Gus is here at G4G11 (his young mind duly warped by Martin). In the Gift Bag, you'll find a bookmark with a pointer to a Virtual Manipulative he is creating.

This paper is meant to supplement a 10 minute talk by the same name at Gathering for Gardner 11 in Atlanta, March, 2014. There were ~30 slides in the presentation.

References

[1] `http://dialectrix.com/langford.html`
[2] Mathematical Games, Scientific American, November and December 1967.
[3] Martin Gardner revisits Langford's Problem in *Mathematical Magic Show*, published by Alfred A. Knopf, ISBN 0-88385-449-X, first and second editions.

[4] Need to Include OEIS sequence numbers.

Topics not included!

Parity / Covering argument for impossibility.
Nickerson and Skolem Variants, where 2nd occurrence of k is in position p+k, or 0's are allowed.
Higher Order Sequences using triplets, quadruplets, etc, instead of pairs.
Steiner Triple System.. relates to Langford's problem.
Ladder graphs
Hooked and Looped sequences.
Sequences with arbitrary sets of pairs.
Order of Complexity, Upper bounds, estimates.
R.O.Davies vs Dave Moore, Scattergood's construction methods.
Animation of algorithm working on n=7?
Solution Tree for n=7
Google Logo
It's not necessarily a "problem"!
Can a generator exist - just emits complete arrangements via some grammar?
Note: I have not seen Martin's Langford Problem folders.

Volume 1 of The Art of Computer Programming [TAOCP] came out in 1968.
Volume 4a (Chapter 7) of TAOCP, Combinatorial Searching, uses Langford's Problem to illustrate!

Dr. Matrix and C. D. Langford were approx same age!
MG introduced IJM when CDL introduced LP!
Dr. Matrix claims to have found the only solution to LP for n=5.
He uses that 10-digit number on his secretly-built Xphone.